Introduction to Combinatorial anabelian geometry

§1 background

§2 semi-graphs of anabelioids and comb GC

§3. proof — tripod case

§4. proof — affine case

$\boxed{\S 1}$ k : NF or MLF $\hookrightarrow \bar{k}$: alg closure

$[k : \mathbb{Q}] < \infty$ or $[k : \mathbb{Q}_p] < \infty$

$X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ $\quad \bar{X} := X \times_k \bar{k}$

absolute Gal gp of k

$\rightsquigarrow 1 \to \pi_1(\bar{X}) \to \pi_1(X) \to G_k \to 1$ (exact)

$\rightsquigarrow G_k \xrightarrow{\rho} \mathrm{Out}(\pi_1(\bar{X}))$ outer Gal rep

$\wr$

$\hat{F}_2$

① Belyi

② Mats

generalizatio

MLF $\hookrightarrow \bar{k}$: alg closure

or $(k : \mathbb{Q}_p) < \infty$

$\{0, 1, \infty\}$ $\overline{X} := X \times_k \bar{k}$

absolute Gal gp of k

$) \to \pi_1(X) \to G_k \to 1$ (exact)

Out$(\pi_1(\overline{X}))$ outer Gal rep

$\cong \hat{F}_2$

binatorial anabelian geometry

$=: $ CAG

d

f anabelioids and comb GC

ripod case

affine case

① Belyi's theorem ρ is injective ($\rightsquigarrow G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$)

generalization

Galois representations on profinite braid gps

"topological proof"

② Matsumoto's thm

If X affine hyperbolic curve/k, then ρ is also injective

generalization

alternative proof

③ Mochizuki's "algebraic proof"

④ Hoshi-Mochizuki's thm If X arbitrary hyperbolic curve/k, then ρ is also inj

Idea of ③ ④

applying the theory of CAG, especially the theory of combinatorial cuspidalization $=: $ CC

What

$\to G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$)

pological proof"

ic curve$/k$,

ue

f"

hyperbolic curve$/k$,

$= C^{C}$

ation

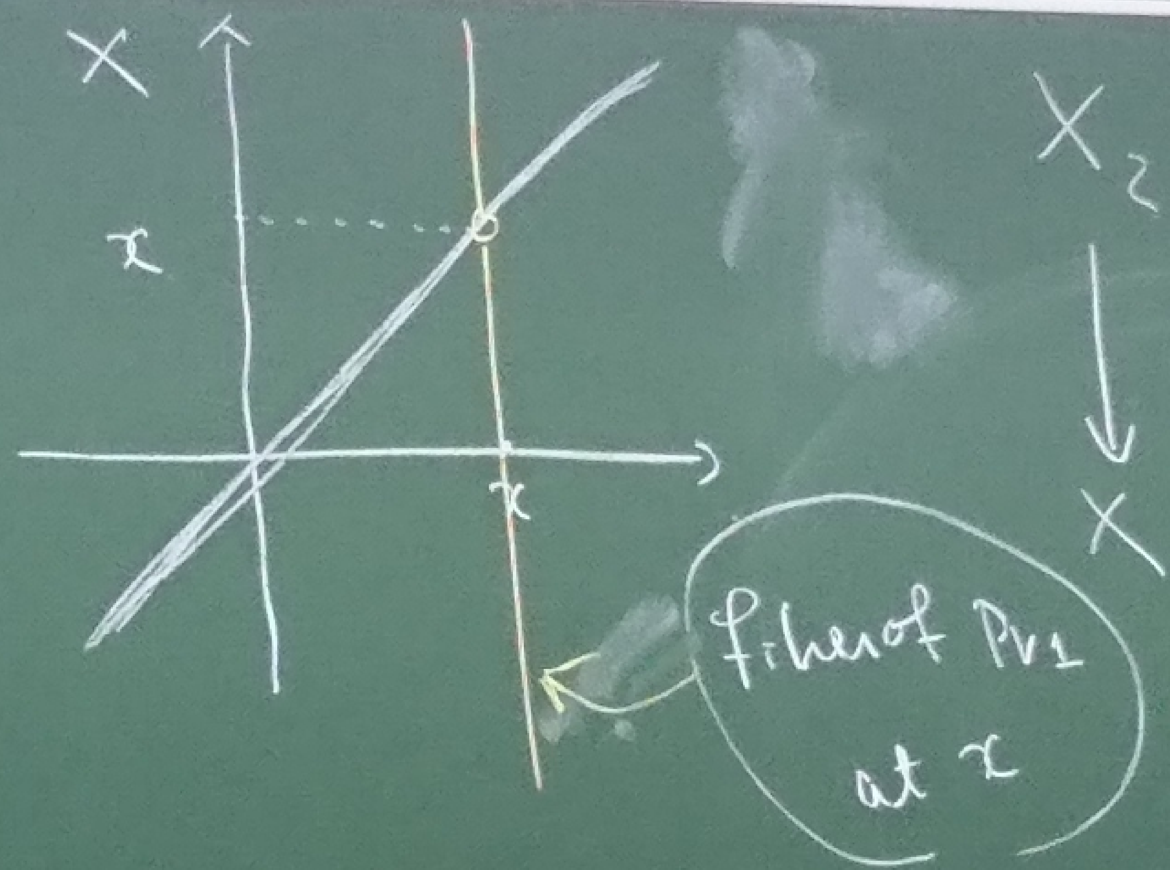
What is cc?

X : hyperbolic curve $/ \bar{k}$ (alg closed field (ch=0))

$X_n := \{ (f_1, \ldots, f_n) \in X^{\times n} \mid f_i \neq f_j \text{ if } i \neq j \}$

$\rightsquigarrow$ n-th configuration space of X

ex ($n=2$)



$X_2 \xrightarrow{pr_1} X$

"family of $X \setminus \{pt\}$"

$X_n \to X_{n-1} \to \cdots \to X_2 \to X$ standard projections

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \mapsto \cdots \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1$

If $\Pi_m := \pi_1(X_m)$, then "standard surjections" induce "standard"

$$\Pi_n \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \cdots\cdots \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1$$

$K_m := \mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_m)$, $\Pi_0 = \{1\}$

$$\rightsquigarrow \{1\} = K_n \subseteq K_{n-1} \subseteq \cdots\cdots \subseteq K_2 \subseteq K_1 \subseteq K_0 = \Pi_n$$

Note that "$K_{m+1} \subseteq K_m$" induces

$$\begin{array}{ccccccccc}
1 \to & K_m/K_{m+1} & \to & \Pi_n/K_{m+1} & \to & \Pi_n/K_m & \to 1 \\
& \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\
1 \to & \pi_1\big((X_{m+1})_{\bar\eta}\big) & \to & \pi_1(X_{m+1}) & \to & \pi_1(X_m) & \to 1
\end{array}$$

hyperbolic curve of type $(g, r+m)$

$\bar\eta$: geom gen pt of X_m

Idea of ③ ④

applying the

especially the

"standard surjections" induce

$\twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1$ "standard"

$\{1\}$

$\subseteq K_1 \subseteq K_0 = \Pi_n$

$/K_m \to 1$

$K_m) \to 1$

$\ldots$ pt of X_m

(4) Hoshi-Mochizuki's thm If X: arbitrary hyperbolic curve$/k$,

then ρ is also inj

Idea of (3) (4)

applying the theory of CAG, $= CC$

especially the theory of combinatorial cuspidalization

- $\mathrm{Aut}^F(\Pi_n) := \{ \alpha \in \mathrm{Aut}(\Pi_n) \mid$ for $\forall$ fiber subgp $J \subseteq \Pi_n$, $\alpha(J) = J \}$

F-admissible

$\mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_{n'})$, $X_n \to X_{n'}$ proj

- $\mathrm{Aut}^C(\Pi_n) := \{ \alpha \in \mathrm{Aut}(\Pi_n) \mid \alpha(K_m) = K_m \ (\forall m)$ and $\alpha : K_m/K_{m+1} \xrightarrow{\sim} K_m/K_{m+1}$ induces bij of cuspidal inertia subgps $\}$

C-admissible

$X_n \to X_{n-1} \to$

$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$

$\mathrm{Aut}^{FC}(\Pi_m$

hyperbolic
curve $/k$,

$\ell = \mathbb{C}\mathbb{C}$

ation

subgp $J \subseteq \Pi_n$

$\Pi_n \twoheadrightarrow \Pi_{n'}$)

$X_n \to X_{n'}$ proj

) and

ces

$\mathrm{Aut}^{FC}(\Pi_n) := \mathrm{Aut}^{F}(\Pi_n) \cap \mathrm{Aut}^{C}(\Pi_n)$

$\mathrm{Out}^{FC}(\Pi_n) := \mathrm{Aut}^{FC}(\Pi_n) / \mathrm{Inn}(\Pi_n)$

Thus, the standard proj $X_{n+1} \to X_n$ induces

$$\boxed{\mathrm{Out}^{FC}(\Pi_{n+1}) \to \mathrm{Out}^{FC}(\Pi_n)}$$

"comb cusp"

Thm (Hoshi-Mochizuki) "main"

> For $n \geq 1$, this hom is injective

pro-ℓ
$\mathrm{Out}^{*}(\Pi_{n+1}) \to \mathrm{Out}^{*}(\Pi_n)$

Rmk (1) In fact, this hom is bijective ($n \geq 4$)

(2) $\exists$ pro-ℓ version of Thm ; Rmk (1)

⟨ Sketch of the proof of ③ ④ ⟩

$$
\begin{tikzcd}
G_k \arrow[r] \arrow[dr, "?"] \arrow[ddr, "?"] \arrow[dddr, "?"] & \mathrm{Out}(\underbrace{\pi_1(\overline{X})}_{=\pi_1}) \quad \underline{\text{inj}}\ ? \\
& \mathrm{Out}^{FC}(\pi_1) \arrow[u, hook] \\
& \mathrm{Out}^{FC}(\pi_2) \arrow[u, "\text{by Thm, inj}"'] \\
& \mathrm{Out}^{FC}(\pi_3) \arrow[u] \arrow[r, "\text{``tripod hom''}"] & \mathrm{Out}^{FC}(\pi_1(\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\}))
\end{tikzcd}
$$

inj !!

In the following, for simplicity,

we consider the case of $\underline{n=1}$

$$\boxed{\mathrm{Out}^{FC}(\pi_2) \to \mathrm{Out}^{FC}(\pi_1) \quad \text{inj}}$$

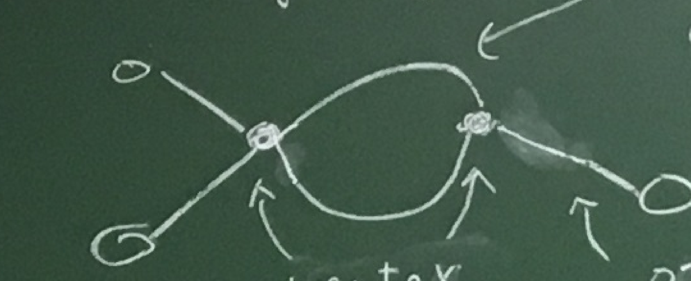
§2

Def

$\boxed{\S 2.}$

Def (1) <u>semi-graph of anabelioids</u> $\mathcal{G}$

consists of the following data:



(i) conn semi-graph $\mathbb{G}$ (Galois category)

(ii) Fr $^\forall v \in \mathbb{V}(\mathbb{G})$, conn anabelioid $\mathcal{G}_v$

← vertices

(iii) Fr $^\forall e \in \mathcal{E}(\mathbb{G})$, conn anabelioid $\mathcal{G}_e$

← edges

(iv) Fr $^\forall v \in \mathbb{V}(\mathbb{G})$, $^\forall e \in \mathcal{E}(\mathbb{G})$ abuts to v, $^\forall b$: branch of e,

a morphism of conn anabelioids $b_{e\to v} : \mathcal{G}_e \to \mathcal{G}_v$

to v, $\forall b$ branch of e,

helioid

ata :

category)

v

(2) X / $\exists$ alg closed field (ch=0) pointed stable curve

$\rightsquigarrow$ $\exists\, \mathcal{G}$: semi-graph of anabelioids of PSC-type

(i) $\mathbb{G}$: dual semi-graph of X

(ii) $\forall v \in V(\mathbb{G}) \longleftrightarrow X_v$: irr component

$$\underset{\text{anabelioid}}{\mathcal{G}_v} := \mathcal{B}\left(X_v \setminus \left((X_{\text{non-sm}} \cup X_{\text{mark}}) \cap X_v \right) \right)$$

(iii) $\forall e \in \mathcal{E}(\mathbb{G})$ abuts to v $\longrightarrow X_v$: irr comp

($\rightsquigarrow$ "e" may be considered as a "cusp" of smooth curve

$X_v \setminus \left((X_{\text{non-sm}} \cup X_{\text{mark}}) \cap X_v \right)$)